BULETINUL Universităţii Petrol – Gaze din Ploieşti	Vol. LVIII No. 2/2006	9 - 16	Seria Matematică - Informatică - Fizică

On the Definition of Multidimensional Generalized Riemann Integral

Mihai Pascu

Universitatea Petrol-Gaze din Ploieşti, Bd. Bucureşti 39, Ploieşti, Catedra de Matematicã
Institutul de Matematică al Academiei Române
e-mail: Mihai.Pascu@imar.ro

Abstract

The paper wishes to investigate the possibilities of defining the multidimensional generalized Riemann integral. We propose a definition which covers the cases of unbounded domain and unbounded function at the same time. The equivalence between generalized integrability and the absolute integrability of functions depending on more than one variable is proved.

Key words: *multidimensional generalized Riemann integral, absolute integrability, Jordan measure*

Introduction

Lebesgue's theory of integration does not make any distinction between the integration of bounded functions on compact domains and the integration of unbounded functions or the integration of functions defined on unbounded domains. But, in the case of Riemann integral, definition of which is more elementary than the definition of Lebesgue integral, the situation changes.

In this paper we shall discuss a definition of the multidimensional generalized (or improper) Riemann integral which will cover at the same time the cases of unbounded functions and of unbounded domains. Always, in this paper, integrable means Riemann integrable.

The definition of the generalized integral given for simple integrals is usually the following: a function $f:[a,b)\to \boldsymbol{R}$, $b\in(a,\infty]$ which is integrable on [*a, y*] for every $y\in(a,b)$ is improper integrable on [*a, b*) if there exists $\lim\limits_{y\to b}\int\limits_a^y f(x)\mathrm{d}x\in\boldsymbol{R}$. If the limit exists, then it is called the generalized or improper integral of *f* on [*a, b*).

So, the generalized integral on [*a, b*) is approximated with integrals on domains limit of which is [*a, b*). The definition from above is quite satisfactory and covers both cases: the case of unbounded functions and the case of unbounded intervals.

But if we deal with multiple integrals the situation is no more so simple. A first question arises: how to choose the approximating domains? Some answers can be found in [2], [3], [5] In [5] the cases of unbounded domains and unbounded functions in the neighborhood of a point are separately treated. For the case of unbounded domains the notion of section of a domain is used.

We shall adopt here a different approach which is related to the one from [1]. The definition given in [1] covers only the case of unbounded domains. We shall modify it in order to cover the case of unbounded function also. The modification is in the spirit of [4]. But the integral which is defined in [4] is not a Riemann integral.

We shall describe now the content of the paper. In a first section we shall shortly recall, for reader's convenience, the definition and some properties of the multiple Riemann integral. Some other properties will be recalled when used. The second section contains the definition of the generalized Riemann integral and of absolute integrability. Criteria of generalized integrability are also given. In the third section we shall prove that for multidimensional integrals the notions of generalized integrability and absolute integrability are equivalent. This fact points out to a significant difference between one dimensional generalized integrals and multidimensional generalized integrals. There are functions depending on one variable which are integrable in a generalized sense, but which are not absolutely integrable. The most known example is perhaps the function

$$f:(0,\infty)\to \boldsymbol{R},\ f(x)=\frac{\sin x}{x}.$$

We have to mention that everything presented in this paper is only an attempt at systematizing things which are well known (even if we did not find in the literature the definition of the generalized integral which is given below).

More practical criteria of proving the generalized integrability and examples will be given in another paper.

The Multidimensional Riemann Integral

We suppose that the reader is familiar with the notions of Jordan measurable set, Jordan measure and Lebesgue negligible set. We shall denote with J_n the family of sets of $\boldsymbol{R}^n$ which are Jordan measurable and with $\lambda(A)$ the Jordan measure of a set A which is Jordan measurable. Let us only recall that the sets which are Jordan measurable are also bounded.

Definition 1. Let $A\subset \boldsymbol{R}^n$ be a set which is Jordan measurable. A Jordan partition of A is a finite family of nonvoid sets which are Jordan measurable, union of which is equal with A and such that the Jordan measure of the intersection of any two different sets of the family is equal with 0. We shall denote with $d(A)$ the set of Jordan decompositions of A.

Definition 2. If

$$\alpha=\left(A_i\right)_{i=1,\ldots,k}\in d(A)$$

is a Jordan partition of a set A, the number

$$\|\alpha\|=\max_{i=1,\ldots,k}\operatorname{diam}(A_i)$$

is called the norm of the partition.

Definition 3. Let $A\in J_n$ and $\alpha=\left(A_i\right)_{i=1,\ldots,k}\in d(A)$. A system of points $\xi=\left(\xi_i\right)_{i=1,\ldots,k}$ such that $\xi_i\in A_i,\ (\forall)i=1,\ldots,k$ is called a family of intermediate points associated to α. We denote with $\xi(\alpha)$ the set of all the systems of intermediate points associated to a Jordan decomposition α.

Definition 4. Let $A\in J_n$, $f:A\to\boldsymbol{R}$ $\alpha=\left(A_i\right)_{i=1,\ldots,k}\in d(A)$ and $\xi\in\xi(\alpha)$. The number

$$\sigma_\alpha(f,\xi)=\sum_{i=1}^{k}f(\xi_i)\lambda(A_i)$$

is called a Riemann sum associated to *f*, α and ξ.

Definition 5. Let $A \in \mathrm{J}_n$, $f : A \to \boldsymbol{R}$. We say that *f* is Riemann integrable (on *A*) if there exists a real number $I = I(f)$ such that $(\forall)\varepsilon > 0, (\exists)\delta > 0$ with

$$\left|\sigma_\alpha(f,\xi) - I\right| < \varepsilon, (\forall)\alpha \in d(A), \|\alpha\| < \delta, (\forall)\xi \in \xi(\alpha).$$

If such a number exists, then it is unique and it is called the Riemann integral of *f* on *A*.

Theorem 1. Let $A \in \mathrm{J}_n$, $A \subset \overline{\mathrm{int}(A)}$, $f : A \to \boldsymbol{R}$, int(*A*) beeing the interior of *A*, *f* Riemann integrable on *A*. Then *f* is bounded.

Definition 6. Let $A \in \mathrm{J}_n$, $f : A \to \boldsymbol{R}$ bounded $\alpha = (A_i)_{i=1,\dots,k} \in d(A)$, $m_i = \inf\limits_{A_i} f$, $M_i = \sup\limits_{A_i} f$.

1. The number

$$S_\alpha(f) = \sum_{i=1}^{k} M_i \lambda(A_i)$$

is called the superior Darboux sum associated to *f* and α.

2. The number

$$s_\alpha(f) = \sum_{i=1}^{k} m_i \lambda(A_i)$$

is called the inferior Darboux sum associated to *f* and α.

Definition 7. Let $A \in \mathrm{J}_n$, $f : A \to \boldsymbol{R}$, *f* bounded.

1. We call the inferior Darboux integral of *f* on *A* the number

$$\underline{\int_A} f(x)\mathrm{d}x = \sup_{\alpha \in d(A)} s_\alpha(f).$$

2. We call the inferior Darboux integral of *f* on *A* the number

$$\overline{\int_A} f(x)\mathrm{d}x = \inf_{\alpha \in d(A)} S_\alpha(f).$$

Theorem 2. Let $A \in \mathrm{J}_n$, $f : A \to \boldsymbol{R}$, *f* bounded. Then the following properties are equivalent:

1. *f* is Riemann integrable;
2. the inferior and the superior Darboux integrals of *f* are equal;
3. $(\forall)\varepsilon > 0, (\exists)\alpha \in d(A)$ such that $S_\alpha(f) - s_\alpha(f) < \varepsilon$;
4. $(\forall)\varepsilon > 0, (\exists)\delta > 0$ such that

$$S_\alpha(f) - s_\alpha(f) < \varepsilon, (\forall)\alpha \in d(A), \|\alpha\| < \delta.$$

The Definition of the Generalized Multidimensional Riemann Integral

Definition 8. A subset *B* of $\boldsymbol{R}^n$ is called locally Jordan measurable if its intersection with any set which is Jordan measurable is also Jordan measurable.

Notation. If *B* is a locally Jordan measurable set, *f* is a real function defined on *B* and *M* is a positive number, we define a new function f_M through the formula:

$$f_M(x) = -M \quad \text{if } f(x) < -M,$$

$$f_M(x) = f(x) \qquad \text{if} -M \le f(x) \le M$$

and

$$f_M(x) = M \qquad \text{if } f(x) > M.$$

Definition 9. Let $B \subset \boldsymbol{R}^n$ be a locally Jordan measurable set, f a real function defined on B. We say that f is integrable in the generalized sense on B if f_M is integrable for any positive M on any Jordan subset of B and if there exists a real number I with the property that

$$(\forall)\varepsilon > 0,\ (\exists)A_\varepsilon \in \mathsf{J}_n, A_\varepsilon \subset B,\ (\exists)M_\varepsilon > 0$$

such that

$$\left|\int_A f_M(x)\mathrm{d}x - I\right| < \varepsilon,\ (\forall)A \in \mathsf{J}_n,\ A_\varepsilon \subset A \subset B,\ (\forall)M \ge M_\varepsilon.$$

Remark. If f is integrable in the generalized sense on B, then a number I which has the property from definition 9 is unique, it is called the generalized or improper integral of f on B and is denoted with $\int_B f(x)\mathrm{d}x$.

Remark. If f takes only nonnegative values and is integrable in the generalized sense on B and if $A_1 \subset A_2 \subset A,\ 0 \le M_1 \le M_2$ then $\int_{A_1} f_{M_1}(x)\mathrm{d}x \le \int_{A_2} f_{M_2}(x)\mathrm{d}x$.

Proposition 1. Let B be a subset of $\boldsymbol{R}^n$ which is locally Jordan measurable, $f : B \to \boldsymbol{R}$ a function which has the property that f_M is integrable for every positive M on every Jordan subset of B. Then f is integrable in a generalized sense if and only if

$$(\forall)\varepsilon > 0,\ (\exists)A_\varepsilon \in \mathsf{J}_n, A_\varepsilon \subset B,\ (\exists)M_\varepsilon > 0$$

such that

$$\left|\int_A f_M(x)\mathrm{d}x - \int_{A'} f_{M'}(x)\mathrm{d}x\right| < \varepsilon,\ (\forall)A, A' \in \mathsf{J}_n,\ A_\varepsilon \subset A \subset B, A_\varepsilon \subset A' \subset B,$$
$$(\forall)M, M' > M_\varepsilon.$$

Proof. "$\Rightarrow$". This implication is trivial.

"$\Leftarrow$". Let A_j, M_j be such that

$$\left|\int_A f_M(x)\mathrm{d}x - \int_{A'} f_{M'}(x)\mathrm{d}x\right| < \frac{1}{j},\ (\forall)A, A' \in \mathsf{J}_n,\ A_j \subset A \subset B, A_j \subset A' \subset B,$$
$$(\forall)M, M' \ge M_j,\ A_j \subset A_{j+1},\ M_j \le M_{j+1}.$$

Then

$$\left(\int_{A_j} f_{M_j}\mathrm{d}x\right)_j$$

is a Cauchy sequence. Let I be its limit and let j_ε be such that

$$\frac{1}{j_\varepsilon} < \frac{\varepsilon}{2}$$

and

$$\left| \int_{A_j} f_{M_j} \mathrm{d}x - I \right| < \frac{\varepsilon}{2}, (\forall) j \geq j_\varepsilon .$$

Let

$$A \in \mathrm{J}_n,\ A_{j_\varepsilon} \subset A \subset B,\ M \geq M_{j_\varepsilon} .$$

We have

$$\left| \int_A f_M(x)\mathrm{d}x - I \right| \leq \left| \int_A f_M(x)\mathrm{d}x - \int_{A_{j_\varepsilon}} f_{M_{j_\varepsilon}}(x)\mathrm{d}x \right| + \left| \int_{A_{j_\varepsilon}} f_{M_{j_\varepsilon}}(x)\mathrm{d}x - I \right| < \varepsilon \quad .$$

Therefore f is integrable in the generalized sense on B and

$$\int_B f(x)\mathrm{d}x = I .$$

We shall give now two other criteria of generalized integrability, applicable to functions which take only nonnegative values. These criteria are similar to the well known boundedness and comparison criteria for functions which depends on only one variable.

Proposition 2. Let B be a locally Jordan measurable subset of $\boldsymbol{R}^n$ $f : B \to [0,\infty)$ a function which has the property that f_M is integrable for every positive M on every Jordan subset of B. Then f is integrable in the generalized sense if and only if there exists a constant $C > 0$ such that

$$\int_A f_M(x)\mathrm{d}x \leq C$$

for every $M > 0$ and for every Jordan measurable subset A of B.

Proof. "$\Rightarrow$". If f is integrable in the generalized sense, then

$$\int_A f_M(x)\mathrm{d}x \leq \int_B f(x)\mathrm{d}x$$

for every $M > 0$ and for every Jordan measurable subset A of B.

"$\Leftarrow$". Let

$$I = \sup\left\{ \int_A f_M(x)\mathrm{d}x;\ A \in \mathrm{J}_n, A \subset B, M > 0 \right\},$$

which, accordingly to our hypothesis is a real (finite) number. Then for every $\varepsilon > 0$ there exists $A_\varepsilon \in \mathrm{J}_n, A_\varepsilon \subset B, M_\varepsilon > 0$ such that

$$I - \int_{A_\varepsilon} f_{M_\varepsilon}(x)\mathrm{d}x < \varepsilon .$$

Since f takes only nonnegative values, we have that

$$I - \int_A f_M(x)\mathrm{d}x < \varepsilon, (\forall) A \in \mathrm{J}_n,\ A_\varepsilon \subset A \subset B, (\forall) M \geq M_\varepsilon ,$$

and the proof is complete.

Proposition 3. Let B be a locally Jordan measurable subset of $\boldsymbol{R}^n$, $f, g : B \to [0,\infty)$ two functions which have the property that f_M , g_M are integrable for every positive M on every Jordan subset of B and

$$f(x) \leq g(x), (\forall) x \in B \setminus N ,$$

where N is a Lebesgue negligible subset of B. Then

i) if g is integrable in the generalized sense, then f is also integrable in the generalized sense;

ii) if f is not integrable in the generalized sense, then g is not integrable in the generalized sense.

Proof. We notice that the hypothesis imply that

$$f_M(x) \le g_M(x),\ (\forall)x \in B \setminus N,\ (\forall)M > 0 .$$

Then the conclusion of the proposition is a consequence of Proposition 2 and of the monotony property of the integral.

Corollary. Let B be a locally Jordan measurable subset of $\boldsymbol{R}^n$ $f : B \to [0,\infty)$ a function which is integrable in the generalized sense, $g : B \to [0,\infty)$ integrable in the generalized sense and bounded. Then fg is integrable in the generalized sense.

Absolutely Integrable Functions

Definition 10. Let $B \subset \boldsymbol{R}^n$ be a locally Jordan measurable set, f a real function defined on B which has the property that f_M is integrable for any positive M on any Jordan subset of B. We say that f is absolutely integrable on B or that the integral of f on B is absolutely convergent if $|f|$ is integrable in the generalized sense on B.

Lemma 1. If f is integrable in the generalized sense on B, then there exists positive constants C and M_0 such that

$$\left|\int_A f_M(x)\mathrm{d}x\right| \le C,\ (\forall)A \in \mathsf{J}_n,\ A \subset B,\ (\forall)M > M_0 .$$

Proof. Accordingly to Definition 9, there exists a set A_0 and a positive number M_0 such that

$$\left|\int_{A'} f_{M'}(x)\mathrm{d}x\right| < 1,\ (\forall)A' \in \mathsf{J}_n,\ A' \subset B \setminus A_0,\ (\forall)M \ge M_0 .$$

We can take

$$C = \int_{A_0} |f(x)|\mathrm{d}x + 1 .$$

Theorem 3. Let $B \subset \boldsymbol{R}^n$ be a locally Jordan measurable set, f a real function defined on B. Then f is absolutely integrable on B if and only if f is integrable in the generalized sense on B.

Proof. "$\Rightarrow$" Let us take an arbitrary $\varepsilon > 0$. Since f is absolutely integrable,

$$(\exists)A_\varepsilon \in \mathsf{J}_n, A_\varepsilon \subset B,\ (\exists)M_\varepsilon > 0$$

such that

$$\left|\int_A |f|_M(x)\mathrm{d}x - \int_{A'} |f|_{M'}(x)\mathrm{d}x\right| < \frac{\varepsilon}{2},\ (\forall)A, A' \in \mathsf{J}_n,\ A_\varepsilon \subset A \subset B, A_\varepsilon \subset A' \subset B,$$

$$(\forall)M, M' > M_\varepsilon .$$

We may assume that $A \subset A', M \le M'$. We have that

$$\left|\int_A f_M(x)\mathrm{d}x - \int_{A'} f_{M'}(x)\mathrm{d}x\right| \le \left|\int_A f_M(x)\mathrm{d}x - \int_{A'} f_M(x)\mathrm{d}x\right| + \left|\int_{A'} f_M(x)\mathrm{d}x - \int_{A'} f_{M'}(x)\mathrm{d}x\right| .$$

From the definition of f_M one can see that

$$\left|f_M(x)-f_{M'}(x)\right| \le |f|_{M'}(x)-|f|_M(x)$$

and that

$$\left|f_M(x)\right| = |f|_M(x)$$

for every x. Therefore

$$\left|\int_A f_M(x)\mathrm{d}x-\int_{A'} f_{M'}(x)\mathrm{d}x\right| \le \left|\int_{A'\setminus A} f_M(x)\mathrm{d}x\right|+\int_{A'}\left|f_M(x)-f_{M'}(x)\right|\mathrm{d}x \le$$

$$\le \int_{A'\setminus A}\left|f_M(x)\right|\mathrm{d}x+\int_{A'}\left(|f|_{M'}(x)-|f|_M(x)\right)\mathrm{d}x=$$

$$=\left|\int_{A'}|f|_M(x)\mathrm{d}x-\int_A|f|_M(x)\mathrm{d}x\right|+\left|\int_{A'}|f|_{M'}(x)\,\mathrm{d}x-\int_{A'}|f|_M(x)\,\mathrm{d}x\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon .$$

We have used here the remark which precedes Proposition 1. Then Proposition 1 implies that f is integrable in the generalized sense.

"$\Leftarrow$" Let

$$f_+=\max(f,0)=\frac{|f|+f}{2}, \text{ and } f_-=\max(-f,0)=\frac{|f|-f}{2}$$

be the positive and negative parts of f. Then

$$f_M=f_{+,M}-f_{-,M},\ |f|_M=f_{+,M}+f_{-,M}$$

for every $M>0$.

Let us suppose that f is not absolutely integrable. Then

$$(\forall)\,j\in \boldsymbol{N}^*,(\exists)A_j\in \mathrm{J}_n,\ M_j>0$$

such that

$$\left|\int_{A_j}|f|_{M_j}(x)\mathrm{d}x\right|>C+2j,\ (\forall)\,j\in \boldsymbol{N}^*,$$

where C is the constant from Lemma 1. Therefore

$$\left|\int_{A_j} f_{+,M_j}(x)\mathrm{d}x+\int_{A_j} f_{-,M_j}(x)\mathrm{d}x\right|>C+2j,\ (\forall)\,j\in \boldsymbol{N}^*.$$

We may assume that $M_j>M_0,\ (\forall)\,j>0$. Hence

$$\left|\int_{A_j} f_{+,M_j}(x)\mathrm{d}x-\int_{A_j} f_{-,M_j}(x)\mathrm{d}x\right|\le C,\ (\forall)\,j\in \boldsymbol{N}^*.$$

One obtains that

$$\left|\int_{A_j} f_{+,M_j}(x)\mathrm{d}x\right|>j,\ (\forall)\,j\in \boldsymbol{N}^*.$$

Since f_{+,M_j} is integrable on A_j, we deduce that there exists a Jordan partition $(A_i^{(j)})_{i=1,\dots,k}$ of A_j such that

$$\sum_{i=1}^{k} m_i^+ \lambda(A_i^{(j)}) \geq j,\ (\forall) j > 0,$$

where $m_i^+ = \inf\limits_{A_i^{(j)}} f_+$. We assume that the set I of indices i which have the property that $m_i^+ > 0$ is equal to $\{1,\ldots,l\}$ for some $l \leq k$ (this assumption does not affect the correctness of the proof). Then

$$\int\limits_{\bigcup\limits_{i=1}^{l} A_i^{(j)}} f_M(x)\mathrm{d}x \geq \sum_{i=1}^{k} m_i^+ \lambda(A_i^{(j)}) \geq j,\ (\forall) j > 0.$$

This is in contradiction with the fact that

$$\left|\int\limits_A f_M(x)\mathrm{d}x\right| \leq C,\ (\forall) A \in \mathrm{J}_n,\ A \subset B,\ (\forall) M > 0.$$

The proof is complete.

References

1. B o b o c , N. - *Analiză Matematică*, partea a II-a, Editura Universității din Bucureşti, Bucureşti, 1998
2. C h i l o v , G. - *Analyse Mathématique, Fonctions de plusieurs variables réelles*, Mir, Moscou, 1975
3. K u d r i a v t z e v , L.D. - *A course in Mathematical Analysis* (russian), Moskva vishaia shkola, 1988
4. L o o m i s , L.H. - *Advanced calculus*, Addison Wesley Reading, Bucureşti, 1968
5. N i c o l e s c u , M., D i n c u l e a n u , N., M a r c u s , S. - *Analiză Matematică*, vol II, second edition, Editura Didactică şi Pedagogică, Bucureşti, 1971
6. P r e c u p a n u , A. - *Analiză Matematică, Funcții reale,* Editura Didactică şi Pedagogică, Bucureşti, 1976

Observații asupra definiției integralei Riemann generalizate

Rezumat

Investigăm modalitățile de definire a integralelor Riemann generalizate (sau improprii) multidimensionale. Propunem o definiție care acoperă în acelaşi timp cazul funcțiilor nemărginite şi cazul domeniilor nemărginite. Este demonstrată echivalența dintre integrabilitatea generalizată şi integrabilitatea absolută a funcțiilor care depind de mai multe variabile.